Counting Ordinary Lines in Complex Space

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Background

Sylvester-Gallai Theorem: Given a finite set of points in $\mathbb{R}^2$, either

1. all the points are collinear; or
2. there is a line passing through exactly 2 points, called an *ordinary* line.

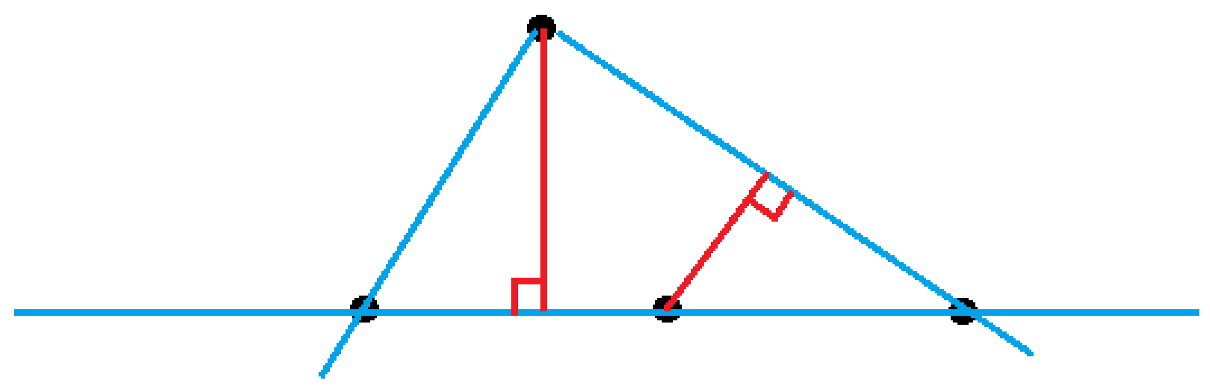

Figure 1: Kelly's [4] picture proof of the Sylvester-Galllai Theorem.

Question: How many ordinary lines do n non-collinear points determine? $t_r :=$ # of lines through exactly r points. (So $t_2 =$ # of ordinary lines.)

Answer (Green-Tao)[2]: n non-collinear points in $\mathbb{R}^2$ have $t_2 \geq \frac{1}{2}n$. This is tight due to the following construction by Böröczky:

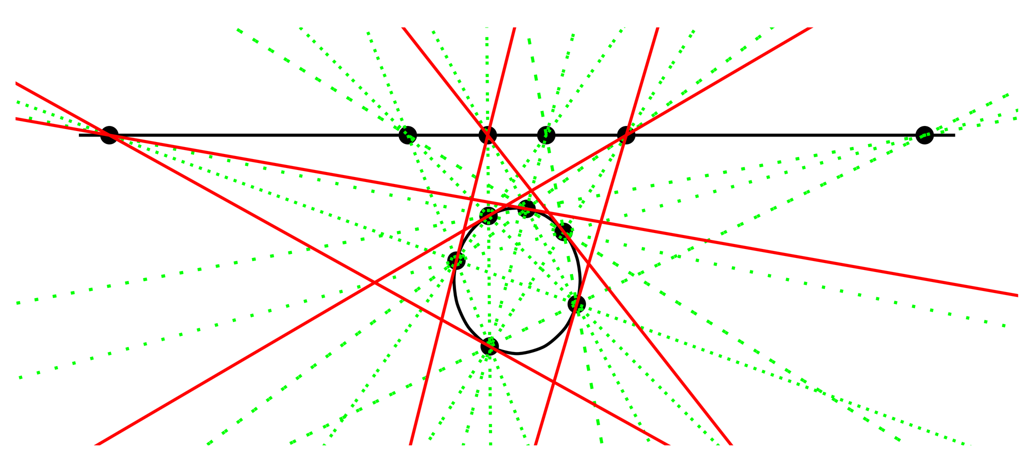

Figure 2: $\frac{n}{2}$ points on a circle and $\frac{n}{2}$ points on a line determining $\frac{n}{2}$ ordinary lines

The Sylvester-Gallai Theorem is not true over $\mathbb{C}^2$. Here is a counterexample:

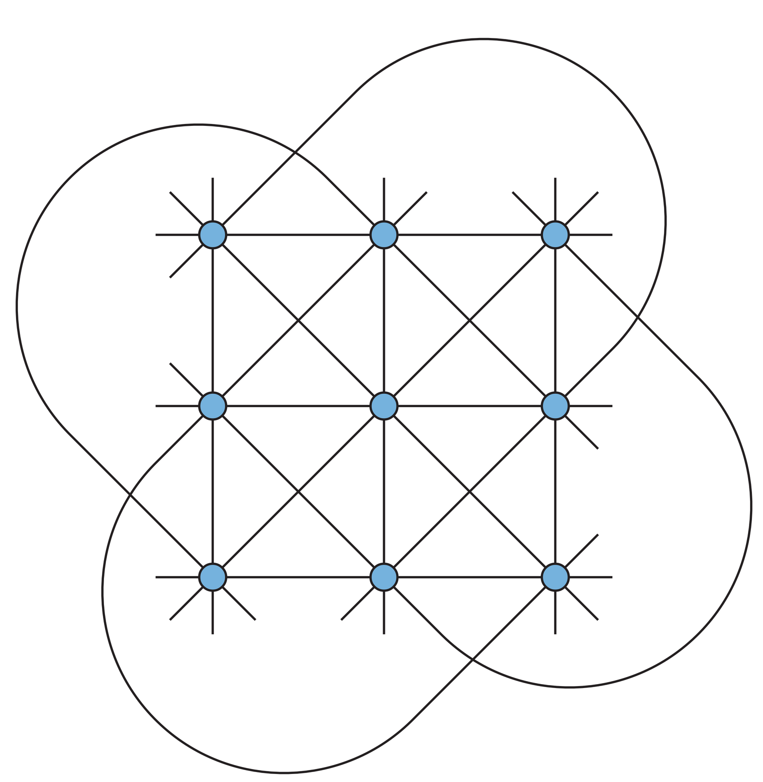

Figure 3: **Hesse configuration[1]:** 9 inflection points of $X^3 + Y^3 + Z^3 = 0$.

We do have an analogue of the Sylvester-Gallai Theorem in $\mathbb{C}^3$:
Theorem (Kelly, [3]) Given a finite set of points in $\mathbb{C}^3$, either:

1. all the points are coplanar; or
2. there exists an ordinary line.

Main Question

Given n points in $\mathbb{C}^3$, not all coplanar, how many ordinary lines do they determine?

Results

Theorem 1: *Given a set of n points in $\mathbb{C}^3$, at most $n-2$ points in any plane,*

$$t_2 \geq \frac{3}{2}n.$$

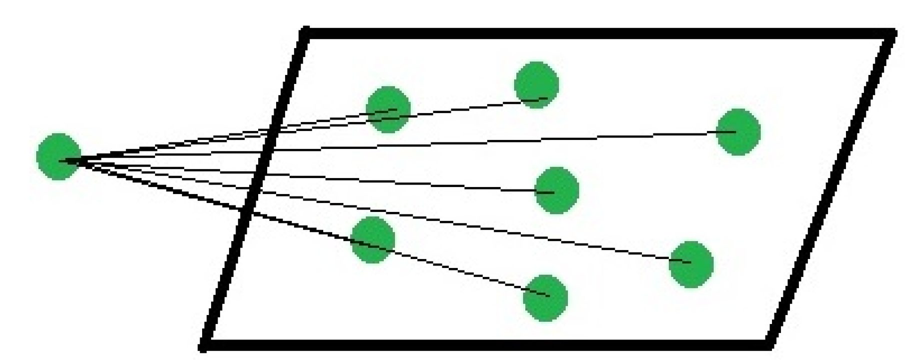

Figure 4: **Exceptional Case:** $n-1$ coplanar points have as few as $n-1$ ordinary lines.

Theorem 2: *Given a set of n points in $\mathbb{C}^3$, at most $\frac{2}{3}n$ points in any plane,*

$$t_2 \geq \frac{3}{2}n + c\sum_{r\geq 4} r^2 t_r.$$

Theorem 3: *Given a set of n points in $\mathbb{C}^4$, at most $\frac{2}{3}n$ in any three-dimensional affine subspace,*

$$t_2 \geq \frac{1}{12}n^2.$$

Proof Sketch

- Given $\{v_1, \ldots, v_n\} \in \mathbb{C}^d$, let V be the $n \times d$ matrix with i^{th} row v_i.
- Construct a matrix A, where each row of A corresponds to coefficients of a collinear triple v_i, v_j, v_k.

$$\overset{A}{\begin{bmatrix} * & * & * & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & * & * & 0 & * & \cdots & 0 \\ & & & \vdots & & & & \\ 0 & * & 0 & 0 & * & 0 & \cdots & * \end{bmatrix}} \overset{V}{\begin{bmatrix} \text{——} & v_1 & \text{——} \\ \text{——} & v_2 & \text{——} \\ & \vdots & \\ \text{——} & v_n & \text{——} \end{bmatrix}} = 0$$

Figure 5: * indicates a nonzero coefficient of a collinear triple.

Proof Sketch Continued

If A has no large zero submatrix:

- If t_2 is small enough, we can show A has high rank.
- This forces V to have small rank, small enough to make the points span only 2 dimensions. Contradiction!

If A has a large zero submatrix:

$$\begin{array}{c|ccc|ccccc|} & & W & & & & Y & & \\ \hline & 0 & 0 & 0 & * & * & * & 0 & 0 \\ U & 0 & 0 & 0 & * & 0 & * & 0 & * \\ & & \vdots & & & & \vdots & & \\ & 0 & 0 & 0 & 0 & * & 0 & * & * \\ \hline & * & 0 & 0 & * & 0 & 0 & * & 0 \\ X & 0 & * & * & 0 & * & 0 & 0 & 0 \\ & * & * & * & 0 & 0 & 0 & 0 & 0 \end{array}$$

Figure 6: The large zero submatrix has support on rows of U and columns of W.

We get a lemma that one of the the following 2 cases holds:

1. (When $|W|$ is large:) $t_2 \geq \frac{|W|}{2}n$, or
2. (When $|W|$ is small:)There is point with least $\frac{2}{3}n - |W|$ ordinary lines.

Iterate this lemma by pruning points with many ordinary lines. Done!

Open Questions

- Is the bound on Theorem 1 tight?
- Can we get a quadratic bound with at most $\frac{2}{3}n$ points in a plane?

References

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